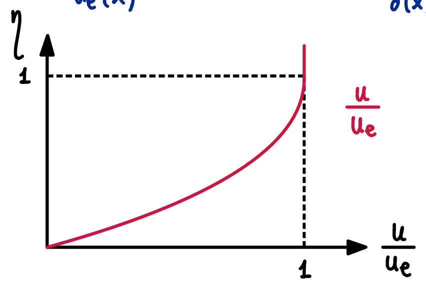
Método de Pohlhausen

Si proponemos una cuártica para $f(\eta)$:

$$\frac{u}{u_e(x)} = f(\eta) \qquad \eta = \frac{y}{\delta(x)}$$

$$f(\eta) = C_0 + C_1\eta + C_2\eta^2 + C_3\eta^3 + C_4\eta^4$$



5 constantes $\longrightarrow$ 5 condiciones : $\begin{cases} \eta = 0 : f|_0 = 0 \; ; \; f''|_0 = -\Lambda \\ \eta = 1 : f|_1 = 1 \; ; \; f'|_1 = f''|_1 = 0 \end{cases}$

POHLHAUSEN SEPARA EL EFECTO DE LA CURVATURA EN LA PARED (Λ) DEL VALOR DE f EN $\eta = 1$: $\boxed{f = f_1 + \Lambda f_2}$

Descomposición :

$$f_1 = \sum_{i=0}^{4} C_{1i}\eta^i$$

$$f_1 = C_{10} + C_{11}\eta + C_{12}\eta^2 + C_{13}\eta^3 + C_{14}\eta^4$$

$$f_1' = C_{11} + 2C_{12}\eta + 3C_{13}\eta^2 + 4C_{14}\eta^3$$

$$f_1'' = 2C_{12} + 6C_{13}\eta + 12C_{14}\eta^2$$

$$\downarrow$$

$$\eta = 0 : f_1 = 0 \; ; \; f_1'' = 0$$

$$\eta = 1 : f_1 = 1 \; ; \; f_1' = 0 \; ; \; f_1'' = 0$$

$$\eta = 0 : \left\{ \begin{array}{l} \boxed{C_{10} = 0} \\ \boxed{C_{12} = 0} \end{array} \right\}$$

$$\boxed{f_1 = 2\eta - 2\eta^3 + \eta^4}$$

$$f_2 = \sum_{i=0}^{4} C_{2i}\eta^i$$

$$f_2 = C_{20} + C_{21}\eta + C_{22}\eta^2 + C_{23}\eta^3 + C_{24}\eta^4$$

$$f_2' = C_{21} + 2C_{22}\eta + 3C_{23}\eta^2 + 4C_{24}\eta^3$$

$$f_2'' = 2C_{22} + 6C_{23}\eta + 12C_{24}\eta^2$$

$$\downarrow$$

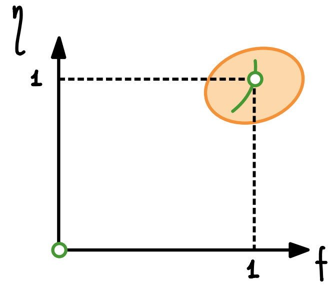
$$\eta = 0 : f_1 = 0 \; ; \; f_1'' = -1$$

$$\eta = 1 : f_1 = 0 \; ; \; f_1' = 0 \; ; \; f_1'' = 0$$

$$\eta = 0 : \left\{ \begin{array}{l} \boxed{C_{20} = 0} \\ \boxed{C_{22} = -\frac{1}{2}} \end{array} \right\}$$

$$\boxed{f_2 = \frac{1}{6}\eta(1-\eta)^3}$$

Zona externa de la capa límite:



$$1-\eta = \epsilon \ (\epsilon \ll 1)$$

$$f_1 = 2\eta - 2\eta^3 + \eta^4 = 2(1-\epsilon) - 2(1-\epsilon)^3 + (1-\epsilon)^4 = 2 - 2\epsilon -$$

$$- 2\left(1 - 3\epsilon + 3\epsilon^2 - \epsilon^3\right) + \left(1 - 2\epsilon + \epsilon^2\right) = 2 - 2\epsilon - 2 + 6\epsilon - 6\epsilon^2 +$$

$$+ 2\epsilon^3 + 1 - 4\epsilon + 6\epsilon^2 - 4\epsilon^3 + \epsilon^4 = 1 - 2\epsilon^3 + \epsilon^4 \longrightarrow \boxed{f_1 \approx 1 + O(\epsilon^3)}$$

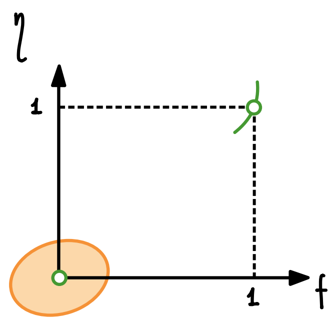
$$f_2 = \frac{1}{6}\eta(1-\eta)^3 = \frac{1}{6}(1-\epsilon)\left[1-(1-\epsilon)\right]^3 = \frac{1}{6}(1-\epsilon)\epsilon^3 = \frac{1}{6}\epsilon^3 - \frac{1}{6}\epsilon^4 \longrightarrow \boxed{f_2 \approx \frac{\epsilon^3}{6} + O(\epsilon^4)}$$

$$\longrightarrow \boxed{\frac{f_2}{f_1} \approx \frac{\epsilon^2}{6} \ll 1}$$

EL EFECTO INTRODUCIDO POR LA CURVATURA NO AFECTA CASI EN ESTA ZONA $\longrightarrow$

$\longrightarrow$ EN POHLHAUSEN LA REGIÓN EXTERIOR DE LA CAPA LÍMITE ES POCO SENSIBLE AL GRADIENTE DE PRESIONES, LO QUE CONLLEVARÁ CIERTO ERROR EN LOS RESULTADOS.

Cerca de la pared:



$$\eta = \epsilon \ (\epsilon \ll 1)$$

$$f_1 = 2\eta - 2\eta^3 + \eta^4 = 2\epsilon - 2\epsilon^3 + \epsilon^4 \longrightarrow \boxed{f_1 \approx 2\epsilon + O(\epsilon^3)}$$

$$f_2 = \frac{1}{6}\eta(1-\eta)^3 = \frac{1}{6}\left(1 - 3\epsilon + 3\epsilon^2 - \epsilon^3\right) = \frac{\epsilon}{6} - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{2} - \frac{\epsilon^4}{6} \longrightarrow \boxed{f_2 \approx \frac{\epsilon}{6} + O(\epsilon^2)}$$

$$\boxed{\frac{f_2}{f_1} \approx \frac{1}{12}}$$

Cerca de la pared, para que el efecto de Λ sea apreciable, debe ser $\Lambda \sim O(10)$: VARIACIONES IMPORTANTES CON RESPECTO A $\Lambda = 0$ (Blasius).

Variaciones de orden unidad tendrían poca influencia cerca de la pared.

$$= \eta\Big|_0^1 - \frac{2+\frac{\Lambda}{6}}{2}\eta^2\Big|_0^1 + \frac{\Lambda}{6}\eta^3\Big|_0^1 + \frac{2-\frac{\Lambda}{2}}{4}\eta^4\Big|_0^1 - \frac{1-\frac{\Lambda}{6}}{5}\eta^5\Big|_0^1 = 1 - 1 - \frac{\Lambda}{12} + \frac{\Lambda}{6} + \frac{1}{2} -$$

$$- \frac{\Lambda}{8} - \frac{1}{5} + \frac{\Lambda}{30} = \frac{3}{10} - \frac{\Lambda}{120} \longrightarrow \boxed{\frac{\delta_1}{\delta} = \frac{1}{10}\left(3 - \frac{\Lambda}{12}\right)}$$

Espesor de cantidad de movimiento:

$$\delta_2 = \delta\int_0^1 f(1-f)\,d\eta \longrightarrow \frac{\delta_2}{\delta} = \underbrace{\int_0^1 f\,d\eta}_{I} - \underbrace{\int_0^1 f^2\,d\eta}_{J}$$

Resolvemos I:

$$I = \int_0^1 f\,d\eta = \int_0^1 d\eta - \int_0^1 (1-f)\,d\eta = 1 - \frac{1}{10}\left(3 - \frac{\Lambda}{12}\right) \longrightarrow \boxed{I = \frac{1}{10}\left(7 + \frac{\Lambda}{12}\right)}$$

Resolvemos J:

$$J = \int_0^1 f^2 d\eta = \int_0^1 (f_1 + \Lambda f_2)^2 d\eta = \int_0^1 \left(f_1^2 + 2\Lambda f_1 f_2 + \Lambda^2 f_2^2\right) d\eta = \underbrace{\int_0^1 f_1^2 d\eta}_{J_1} + \underbrace{2\Lambda\int_0^1 f_1 f_2\, d\eta}_{J_2} + \underbrace{\Lambda^2\int_0^1 f_2^2 d\eta}_{J_3}$$

$$J_1 = \int_0^1 f_1^2 d\eta = \int_0^1 (2\eta - 2\eta^3 + \eta^4)(2\eta - 2\eta^3 + \eta^4)\,d\eta = \int_0^1 (4\eta^2 - 4\eta^4 + 2\eta^5 - 4\eta^4 + 4\eta^6 - 2\eta^7 + 2\eta^5 -$$

$$- 2\eta^7 + \eta^8)\,d\eta = \frac{4}{3} - \frac{4}{5} + \frac{1}{3} - \frac{4}{5} + \frac{4}{7} - \frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{9} = \frac{367}{630} \longrightarrow \boxed{J_1 = \frac{367}{630}}$$

$$J_2 = 2\Lambda\int_0^1 f_1 f_2\, d\eta = 2\Lambda\int_0^1 (2\eta - 2\eta^3 + \eta^4)\frac{1}{6}\eta(1-\eta)^3 d\eta \xrightarrow[\dots]{\text{CASIO}} \boxed{J_2 = \frac{71}{7560}\Lambda}$$

$$J_3 = \Lambda^2\int_0^1 f_2^2 d\eta = \Lambda^2\int_0^1 \frac{1}{36}\eta^2(1-\eta)^6 d\eta = \xrightarrow[\dots]{\text{CASIO}} \boxed{J_3 = \frac{1}{9072}\Lambda^2}$$

$$\longrightarrow \frac{\delta_2}{\delta} = \frac{\cdots}{63}\left[\frac{\cdots}{5} - \frac{\cdots}{5}\frac{\Lambda}{12} - \left(\frac{\Lambda}{12}\right)^2\right]$$

Factor de forma :

$$H_{12} = \frac{\delta_1}{\delta_2} = \frac{\frac{\delta_1}{\delta}}{\frac{\delta_2}{\delta}} = \frac{\frac{1}{10}\left(3-\frac{\Lambda}{12}\right)}{\frac{1}{63}\left[\frac{37}{5}-\frac{4}{5}\frac{\Lambda}{12}-\left(\frac{\Lambda}{12}\right)^2\right]} \longrightarrow \boxed{H_{12} = \frac{63}{10}\,\frac{3-\frac{\Lambda}{12}}{\frac{37}{5}-\frac{4}{5}\frac{\Lambda}{12}-\left(\frac{\Lambda}{12}\right)^2}}$$

Coeficiente de fricción :

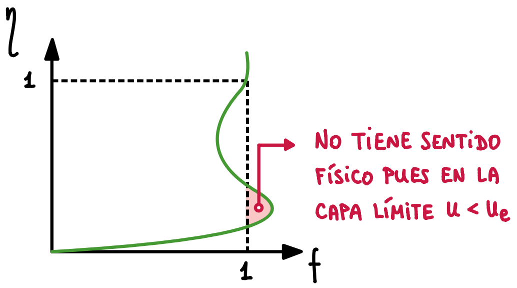
$$\frac{C_f}{2} = \frac{\nu}{U_e\,\delta}\,\frac{df}{d\eta}\bigg|_{\eta=0} = \frac{\nu}{U_e\,\delta}\,\frac{d}{d\eta}\left[2\eta-2\eta^3+\eta^4+\Lambda\,\frac{1}{6}\,\eta(1-\eta)^3\right]_{\eta=0} = \frac{\nu}{U_e\,\delta}\left[2-6\eta+\right.$$

$$\left.+4\eta^3-\frac{\Lambda}{2}\,\eta(1-\eta)^2+\frac{\Lambda}{6}(1-\eta)^3\right]_{\eta=0} = \frac{\nu}{U_e\,\delta}\left(2+\frac{\Lambda}{6}\right) \longrightarrow \boxed{\frac{C_f}{2} = \frac{2\nu}{U_e\,\delta}\left(1+\frac{\Lambda}{12}\right)}$$

Si $\Lambda = -12$: $C_f = 0$ (SEPARACIÓN EN POHLHAUSEN) $\longrightarrow$ $\Lambda \geq -12$ CONDICIÓN NECESARIA

Pohlhausen no sirve si $\Lambda < -12$ porque implicaría curvatura negativa de la pared $\longrightarrow$ flujo reverso

Para $\Lambda > 12$ $\exists\, \eta^* / f(\eta^*) > 1$:



$\longrightarrow \Lambda \leq 12 \longrightarrow -12 \leq \Lambda \leq 12 \longrightarrow \boxed{-1 \leq \frac{\Lambda}{12} \leq 1}$

$\Updownarrow$

$\boxed{-0'1567 \leq \lambda \leq 0'0948}$

Hay otro parámetro interesante : $\boxed{\lambda = \frac{\delta_2^2\,\frac{dU_e}{dx}}{\nu}}$

$$\lambda = \left(\frac{\delta_2}{\delta}\right)^2 \underbrace{\frac{dU_e}{dx}\,\frac{\delta^2}{\nu}} = \frac{\Lambda}{3969}\left[\frac{37}{5}-\frac{4}{5}\frac{\Lambda}{12}-\left(\frac{\Lambda}{12}\right)^2\right]^2 \longrightarrow \boxed{\lambda = \frac{\Lambda}{3969}\left[\frac{37}{5}-\frac{4}{5}\frac{\Lambda}{12}-\left(\frac{\Lambda}{12}\right)^2\right]^2}$$

y entonces :

$\delta_1 = \frac{3}{10}\delta$, $\delta_2 = \frac{37}{315}\delta$, $\frac{C_f}{2} = \frac{2\nu}{U_e\,\delta}$

Ecuación de Karman:

$$\frac{d\delta_2}{dx} + \underbrace{(2+H_{12})\frac{1}{U_e}\frac{dU_e}{dx}\delta_2}_{0} = \frac{C_f}{2} \longrightarrow \frac{37}{315}\frac{d\delta}{dx} = \frac{2\nu}{U_e\,\delta} \longrightarrow \int_0^{\delta}\tilde{\delta}\,d\tilde{\delta} = \int_0^{x}\boxed{\frac{630}{37}\frac{\nu}{U_e}}\,d\tilde{x} \xrightarrow{\delta(0)=0}$$

U_e cte

$$\longrightarrow \frac{\delta^2}{2} = \frac{630}{37}\frac{\nu}{U_e}x \longrightarrow \boxed{\frac{\delta}{x} = 6\sqrt{\frac{35}{37}}\,Re_x^{-1/2}}$$

Con $\frac{\delta}{x}$ podemos obtener $\frac{\delta_1}{x}$, $\frac{\delta_2}{x}$ y $\frac{C_f}{2}$:

$$\frac{\delta_1}{x} = \frac{3}{10}\frac{\delta}{x}$$

$$\boxed{\frac{\delta_1}{x} = \frac{9}{5}\sqrt{\frac{35}{37}}\,Re_x^{-1/2} \approx 1'751\,Re_x^{-1/2}}$$

En la solución exacta de Blasius era $1'721\,Re_x^{-1/2} \longrightarrow 1'7\,\%$ de error

$$\frac{\delta_2}{x} = \frac{37}{315}\frac{\delta}{x}$$

$$\boxed{\frac{\delta_2}{x} = \frac{74}{105}\sqrt{\frac{35}{37}}\,Re_x^{-1/2} \approx 0'685\,Re_x^{-1/2}}$$

En la solución exacta de Blasius era $0'664\,Re_x^{-1/2} \longrightarrow 3'2\,\%$ de error

Karman

$$\frac{C_f}{2} = \frac{d\delta_2}{dx} = \frac{d}{dx}\left(\frac{74}{105}\sqrt{\frac{35}{37}}\sqrt{\frac{\nu x}{U_e}}\right) = $$

$$= \frac{74}{105}\sqrt{\frac{35}{37}}\sqrt{\frac{\nu}{U_e}}\frac{1}{2\sqrt{x}}$$

$$\boxed{\frac{C_f}{2} = \frac{74}{105}\sqrt{\frac{35}{37}}\,Re_x^{-1/2} \approx 0'685\,Re_x^{-1/2}}$$

Blasius: $0'664\,Re_x^{-1/2}$ ($3'2\,\%$ error)

A la vista de los errores, Pohlhausen es una buena aproximación de la solución de Blasius.

VALIDACIÓN CON FALKNER - SKAN

Por último, vamos a ver cómo de bueno es este método para capa límite con gradiente de presiones.

Ecuación de Karman:

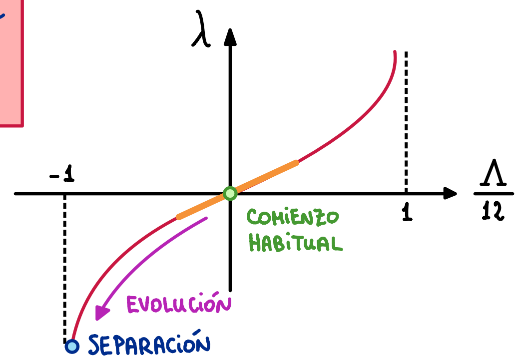
$$\frac{d\delta_2}{dx} + (2+H_{12})\frac{1}{U_e}\frac{dU_e}{dx}\delta_2 = \frac{C_f}{2} \xrightarrow{\cdot\, U_e\frac{\delta_2}{\nu}} \frac{U_e\,\delta_2}{\nu}\frac{d\delta_2}{dx} + (2+H_{12})\lambda = \frac{C_f}{2}\frac{U_e}{\nu}\delta_2 \xrightarrow{\frac{C_f}{2} = \frac{2\nu}{U_e\delta}\left(1+\frac{\Lambda}{12}\right)}$$

$$\longrightarrow \frac{U_e}{\nu}\frac{d\left(\frac{\delta_2^2}{2}\right)}{dx} + (2+H_{12})\lambda = \underbrace{\frac{\delta_2}{\delta}\cdot 2\left(1+\frac{\Lambda}{12}\right)}_{T(\lambda)} \xrightarrow{\frac{\delta_2^2}{\nu} = \frac{\lambda}{dU_e/dx}} \underbrace{U_e\frac{d}{dx}\left(\frac{\lambda}{dU_e/dx}\right)}_{F(\lambda)} = 2\left[T(\lambda) - (2+H_{12})\lambda\right]$$

Por tanto:

$$\frac{C_f}{2} = \frac{2\nu}{U_e\delta}\left(1+\frac{\Lambda}{12}\right)$$

Dado que $\frac{\lambda}{\Lambda} = \frac{\delta}{\delta_2} \longrightarrow \boxed{\lambda = \frac{\Lambda}{63^2}\left[\frac{37}{5} - \frac{4}{5}\frac{\Lambda}{12} - \left(\frac{\Lambda}{12}\right)^2\right]^2}$



Si tenemos una aceleración/deceleración moderada $(\Lambda \sim 1)$, la relación entre λ y Λ se hace lineal: $\boxed{\lambda \approx \left(\frac{37}{315}\right)^2 \Lambda}$

Al igual que analizamos la separación, también analizamos la presencia de un punto de remanso $(u_e = 0)$.

Si $u_e = 0$ no podemos arrancar la solución, y para solucionarlo recurrimos a la solución de Falkner-Skan:

$$u_e = A x^m \quad (m > 0) \longrightarrow \frac{du_e}{dx} = m A x^{m-1} \longrightarrow \frac{1}{du_e/dx} = \frac{1}{m}\frac{x^{1-m}}{A}$$

$$A x^m \frac{d}{dx}\left(\frac{\lambda}{mA} x^{1-m}\right) = F(\lambda)$$

$$x^m \frac{d}{dx}\left(\lambda x^{1-m}\right) = m F(\lambda)$$

$$x^m \frac{d\lambda}{dx} x^{1-m} + x^m \lambda (1-m) x^{-m} = m F(\lambda)$$

$$\boxed{x \frac{d\lambda}{dx} + \lambda (1-m) = m F(\lambda)}$$

LA EVOLUCIÓN DE λ SIGUE ESTA ECUACIÓN DIFERENCIAL

NOTA

$$\lambda = \frac{\delta_2^2 \frac{du_e}{dx}}{\nu}$$

Falkner-Skan: $\begin{cases} \frac{\delta_2}{x} = cte(m) \cdot Re_x^{-1/2} \\ u_e = A x^m \,;\ \frac{du_e}{dx} = \frac{m u_e}{x} \end{cases}$

$$\lambda = \frac{\widetilde{cte}(m)\, x^2}{\nu\, Re_x} \frac{du_e}{dx} = \widetilde{cte}(m) \frac{x^2}{\nu \frac{u_e x}{\nu}} \frac{m u_e}{x} =$$

$$= m\, \widetilde{cte}(m) \longrightarrow \boxed{\lambda = cte(m)}$$

POHLHAUSEN ES CONSISTENTE CON FALKNER-SKAN

$x = x_0 : \lambda = \lambda_{PR}$

¡OJO! ESTAR CERCA DEL PR $(u_e = 0)$ NO ES LO MISMO QUE FLUJO DE PR $(u_e = Ax)$.

En cuanto a la separación: $\lambda_{S-P} \overset{\frac{\Lambda}{12} = -1}{=} -0'1567$ (Pohlhausen) ; $\lambda_{S-FS} = -0'067$ (Falkner - Skan)

EL ERROR ES MUY GRANDE EN LA ZONA DE SEPARACIÓN PORQUE EL MÉTODO DE POHLHAUSEN ES POCO SENSIBLE AL GRADIENTE DE PRESIONES DEL FLUJO EXTERIOR (DEMASIADO OPTIMISTA)

Tabla resumen de resultados:

$\frac{\Lambda}{12}$	λ_P	λ_{FS}	$T(\lambda)$	$H_{12}(\lambda)$	$F(\lambda)$
-1	$-0'1567$	$-0'067$	0	$3'5$	$1'72$
0	0	0	$0'2349$	$2'55$	$0'47$
1	$0'0948$		$0'3556$	$2'25$	$-0'0948$

SEPARACIÓN POHLHAUSEN

BLASIUS

MÁXIMA ACELERACIÓN COMPATIBLE CON QUE NO HAYA SOBREVELOCIDADES DENTRO DE LA CAPA LÍMITE.

NO CAMBIA DEMASIADO, ESPECIALMENTE EN LA ZONA DE ACELERACIÓN $\left(\frac{du_e}{dx} > 0\right.$, es decir, $\left.\lambda > 0\right)$.